PML Implementation for the TLM Propagation Model in acoustics

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IFSTTAR

1. **The Transmission Line Matrix (TLM) method**
   - TLM principle for homogeneous and non-dissipative media
   - TLM principle for heterogeneous and dissipative media
   - Brief review of the absorbing conditions for the TLM method

2. **Adaptation of the PML to the TLM method**
   - The PML for acoustics
   - Approximates PML formulations for the TLM method
   - Assessment of the absorbing layers
   - Numerical results

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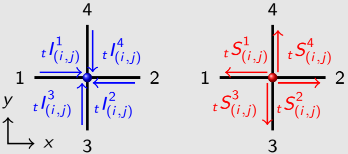
General TLM principle

Huygens principle

Every point of a wave front may be considered the source of secondary wavelets that spread out in all directions with a speed equal to the speed of propagation of the waves.

TLM variables for 2D homogeneous and non-dissipative media



Incident $_tI^n_{(i,j)}$ and scattered $_tS^m_{(i,j)}$ pulses at the node (i,j) and at the time t for an homogeneous and non dissipative medium.

- Incident and scattered pulses:

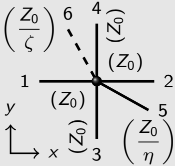
$$_t\mathbf{I}_{(i,j)} = [\ _tI^1;\ _tI^2;\ _tI^3;\ _tI^4]^T_{(i,j)}$$

$$_t\mathbf{S}_{(i,j)} = [\ _tS^1;\ _tS^2;\ _tS^3;\ _tS^4]^T_{(i,j)}$$

- Scattering process:

$$_t\mathbf{S}_{(i,j)} = \ _t\mathbf{D}_{(i,j)}\ _t\mathbf{I}_{(i,j)}$$

TLM for heterogeneous and dissipative media



Additional branches for acoustic propagation in an heterogeneous and dissipative medium (2D).

- Incident and scattered pulses:

$$_t\mathbf{I}_{(i,j)} = [\ _tI^1;\ _tI^2;\ _tI^3;\ _tI^4;\ _tI^5]^T_{(i,j)}$$

$$_t\mathbf{S}_{(i,j)} = [\ _tS^1;\ _tS^2;\ _tS^3;\ _tS^4;\ _tS^5]^T_{(i,j)}$$

- Scattering process:

$$\mathbf{S} = \mathbf{D}(\eta, \zeta).\mathbf{I}$$

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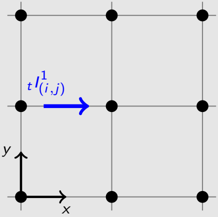
Scattering process

Scattering process for 2D transmission line network



$$\mathcal{R} = \frac{Z_T - Z_L}{Z_T + Z_L}$$

$$\mathcal{T} = 1 + \mathcal{R} = \frac{2Z_T}{Z_T + Z_L}$$

- Z_T impedance of the termination,
- Z_L impedance of the line.

- The scattering matrix:

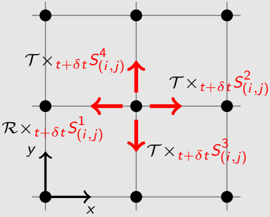
$$_t\mathbf{D}_{(i,j)} = \frac{1}{2}\ _t\begin{bmatrix} \mathcal{R} & \mathcal{T} & \mathcal{T} & \mathcal{T} \\ \mathcal{T} & \mathcal{R} & \mathcal{T} & \mathcal{T} \\ \mathcal{T} & \mathcal{T} & \mathcal{R} & \mathcal{T} \\ \mathcal{T} & \mathcal{T} & \mathcal{T} & \mathcal{R} \end{bmatrix}_{(i,j)} . \tag{1}$$

- Scattering relation:

$$_t\mathbf{S}_{(i,j)} = \ _t\mathbf{D}_{(i,j)}\ _t\mathbf{I}_{(i,j)} , \tag{2}$$

Scattering process

Scattering process for 2D transmission line network



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TLM for heterogeneous and dissipative media

Acoustic pressure

$$ {}_{t}p_{(i,j)} = \frac{2}{\eta_{(i,j)} + \zeta_{(i,j)} + 4} \left[\sum_{n=1}^{4} {}_{t}I^{n}_{(i,j)} + \eta_{(i,j)} \; {}_{t}I^{5}_{(i,j)} \right] $$

Connexion laws

$$ \begin{aligned} {}_{t+\delta t}I^{1}_{(i,j)} &= {}_{t}S^{2}_{(i-1,j)}, \\ {}_{t+\delta t}I^{2}_{(i,j)} &= {}_{t}S^{1}_{(i+1,j)}, \\ {}_{t+\delta t}I^{3}_{(i,j)} &= {}_{t}S^{4}_{(i,j-1)}, \\ {}_{t+\delta t}I^{4}_{(i,j)} &= {}_{t}S^{3}_{(i,j+1)}, \\ {}_{t+\delta t}I^{5}_{(i,j)} &= {}_{t}S^{5}_{(i,j)}. \end{aligned} $$

TLM propagation scheme and wave equation

- TLM scheme for heterogeneous and dissipative network:

$$ {}_{t+\delta t}p_{(i,j)} = \frac{2}{\eta_{(i,j)} + \zeta_{(i,j)} + 4} \left[{}_{t}p_{(i+1,j)} + {}_{t}p_{(i-1,j)} + {}_{t}p_{(i,j+1)} + {}_{t}p_{(i,j-1)} + \eta_{(i,j)} \; {}_{t}p_{(i,j)} \right] - \frac{\eta_{(i,j)} - \zeta_{(i,j)} + 4}{\eta_{(i,j)} + \zeta_{(i,j)} + 4} \; {}_{t-\delta t}p_{(i,j)} $$

- Wave equation:

$$ \left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{\eta + 4}{2} \frac{\delta t^2}{\delta l^2} \frac{\partial^2}{\partial t^2} - \zeta \frac{\delta t}{\delta l^2} \frac{\partial}{\partial t} \right] {}_{t}p_{(i,j)} = 0 $$

- Celerity correction:

$$ c_{\mathrm{TLM}} = \sqrt{\frac{2}{\eta + 4}} \; c, \quad \text{with } c = \frac{\delta l}{\delta t} $$

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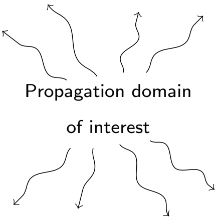
Absorbing conditions for the TLM method

Boundary conditions

Absorbing layers



Ideal infinite propagation domain.

[1] Hofmann *et al.*, *Simulation of outdoor sound propagation with a transmission line matrix method*,Appl. Acoust., 2007.
[2] El-Masri *et al.*, *Vocal tract acoustics using the transmission line matrix (TLM) method*, ICSLP, 1996.
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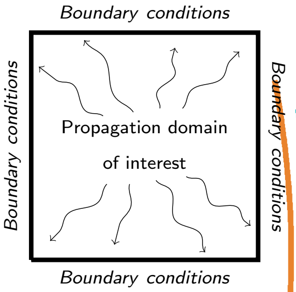
[4] Kagawa *et al.*, *Discrete Huygen's model approach to sound wave propagation*, J. Sound Vib., 1998.
[5] de Cogan *et al.*, *Transmission Line Matrix in Computational Mechanics*, Taylor and Francis, 2006.
[6] Guillaume, *Application of the TLM method for acoustic simulation in urban area*, Phd Thesis, 2009.

Absorbing conditions for the TLM method

Boundary conditions

- Real impedance boundary condition [1],
- Boundary operators [2, 3].

Absorbing layers



Finite propagation domain surrounded by a boundary condition.

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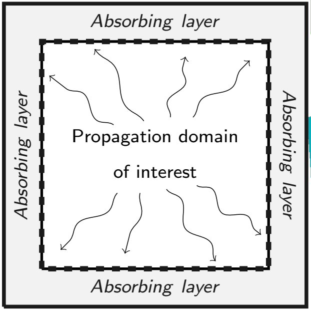
Absorbing conditions for the TLM method

Boundary conditions

- Real impedance boundary condition [1],
- Boundary operators [2, 3].

Absorbing layers

- Dissipative scattering matrix [4, 1],
- Matched connexion laws [5, 6],
- Perfectly matched layer (PML)
  ⇒ **only** through **FDTD calculation** [3].



Finite propagation domain surrounded by an absorbing layer.

Still no rigorous PML implementation for the TLM method in acoustics.

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The PML for acoustics in 2D Cartesian coordinate system

- Mass conservation:

$$\frac{\partial g(x,t)}{\partial t} + c_0^2 \frac{\partial v_x}{\partial x} + \sigma_x \, g(x,t) = 0 \,,$$

$$\frac{\partial h(y,t)}{\partial t} + c_0^2 \frac{\partial v_y}{\partial y} + \sigma_y \, h(y,t) = 0 \,,$$

- Momentum equations:

$$\frac{\partial v_x}{\partial t} + \frac{\partial p}{\partial x} + \sigma_x \, v_x = 0 \,,$$

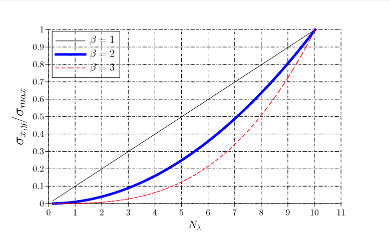
$$\frac{\partial v_y}{\partial t} + \frac{\partial p}{\partial y} + \sigma_y \, v_y = 0 \,,$$

where the acoustic pressure is split:
$p(x,y,t) = g(x,t) + h(y,t)$,

and $\sigma = \sigma_x + \sigma_y$.

$$\boxed{\frac{\partial^2 p(x,y,t)}{\partial t^2} - c_0^2 \left(\frac{\partial^2 p(x,y,t)}{\partial x^2} + \frac{\partial^2 p(x,y,t)}{\partial y^2} \right) + 2\sigma \frac{\partial p(x,y,t)}{\partial t} + \sigma^2 p(x,y,t) = 0}$$

$$\sigma_{(i,j)} = \sigma_{max} \left(\frac{e_{AL} - x_{(i,j)}}{e_{AL}} \right)^{\beta}$$



Identification of the TLM and PML schemes

- PML scheme discretized with the 1[st] order centered finite differences:

$$
{}_{t+\delta t}p_{(i,j)} = \frac{1}{2}\,\frac{1}{(1+\sigma_{(i,j)}\delta t)}\left[\,{}_tp_{(i+1,j)} + {}_tp_{(i-1,j)} + {}_tp_{(i,j+1)} + {}_tp_{(i,j-1)} - 2\sigma^2\delta t^2\;{}_tp_{(i,j)}\right] - \frac{1-\sigma_{(i,j)}\delta t}{1+\sigma_{(i,j)}\delta t}\;{}_{t-\delta t}p_{(i,j)}
$$

- TLM scheme for heterogeneous and dissipative network:

$$
{}_{t+\delta t}p_{(i,j)} = \frac{2}{\eta_{(i,j)}+\zeta_{(i,j)}+4}\left[\,{}_tp_{(i+1,j)} + {}_tp_{(i-1,j)} + {}_tp_{(i,j+1)} + {}_tp_{(i,j-1)} + \eta_{(i,j)}\;{}_tp_{(i,j)}\right] - \frac{\eta_{(i,j)}-\zeta_{(i,j)}+4}{\eta_{(i,j)}+\zeta_{(i,j)}+4}\;{}_{t-\delta t}p_{(i,j)}
$$

Terms identification

$$
\frac{2}{\eta+\zeta+4} = \frac{1}{2}\,\frac{1}{1+\sigma\delta t}\,, \tag{5a}
$$

$$
\frac{2\eta}{\eta+\zeta+4} = -\frac{\sigma^2\delta t^2}{1+\sigma\delta t}\,, \tag{5b}
$$

$$
\frac{\eta-\zeta+4}{\eta+\zeta+4} = \frac{1-\sigma\delta t}{1+\sigma\delta t}\,, \tag{5c}
$$

The three approximate PML formulations

First set of solutions from eqs. (5a) and (5b)

$$\eta = -2\sigma^2\delta t^2$$

$$\zeta = 4\sigma\delta t - 2\sigma^2\delta t^2$$

Second set of solutions from eqs. (5b) and (5c)

$$\eta = -\frac{4\sigma^2\delta t^2}{2+\sigma^2\delta t^2}$$

$$\zeta = \frac{8\sigma\delta t}{2+\sigma^2\delta t^2}$$

Third set of solutions from eqs. (5a) and (5c)

$$\eta = 0$$

$$\zeta = 4\sigma\delta t$$

The three approximate PML formulations

First set of solutions from eqs. (5a) and (5b)

$$\eta = -2\sigma^2\delta t^2$$

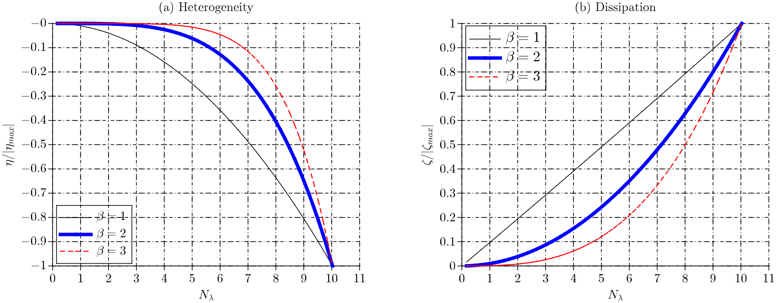
$$\zeta = 4\sigma\delta t - 2\sigma^2\delta t^2$$

Second set of solutions from eqs. (5b) and (5c)

$$\eta = -\frac{4\sigma^2\delta t^2}{2+\sigma^2\delta t^2}$$

$$\zeta = \frac{8\sigma\delta t}{2+\sigma^2\delta t^2}$$

Proportional evolution of η and ζ



(a) Heterogeneity and (b) dissipation terms as a function of the absorbing layer thickness N_λ, for three polynomial gradings.

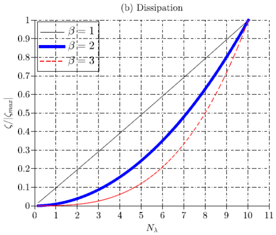
The three approximate PML formulations

Third set of solutions from eqs. (5a) and (5c)

$$\eta = 0$$

$$\zeta = 4\sigma\delta t$$

no heterogeneity $\Rightarrow$ purely dissipative



This set of solutions is equivalent to purely dissipative TLM network.

- TLM dissipative propagation scheme:

$$ {}_{t+\delta t}p_{(i,j)} = \frac{1}{2}\,\frac{1}{1+\sigma\delta t}\left[\,{}_{t}p_{(i+1,j)} + {}_{t}p_{(i-1,j)} + {}_{t}p_{(i,j+1)} + {}_{t}p_{(i,j-1)}\right] - \frac{1-\sigma\delta t}{1+\sigma\delta t}\,{}_{t-\delta t}p_{(i,j)}. $$

- Lossy wave equation:

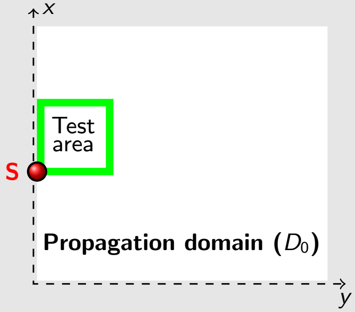
$$ \frac{\partial^2 p(x,y,t)}{\partial t^2} - c_0^2\left(\frac{\partial^2 p(x,y,t)}{\partial x^2} + \frac{\partial^2 p(x,y,t)}{\partial y^2}\right) + 2\sigma\frac{\partial p(x,y,t)}{\partial t} = 0. $$

Assessment of the absorbing layers

Mean error level

$$\text{error}(x,y) = 10\log_{10}\frac{\sum_{t=0}^{T}|p_{\text{ff}}(x,y,t)-p(x,y,t)|^2}{\sum_{t=0}^{T}|p_{\text{ff}}(x,y,t)|^2}$$

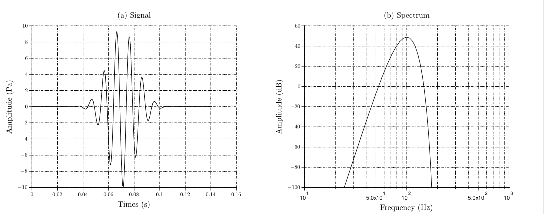
Geometrical configuration of the numerical simulations



Computational domain implemented to assess the performances of an absorbing layer inside the test area (green square).

Source signal

$$p_S(x,y,t) = 10\sin(2\pi ft)\times w_{kr}$$



Gaussian impulse signal (a) ; signal spectrum (b).

Assessment of the absorbing layers

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$$\text{error}(x, y) = 10 \log_{10} \frac{\sum_{t=0}^{T} |\mathbf{p}_{\text{ff}}(x, y, t) - p(x, y, t)|^2}{\sum_{t=0}^{T} |\mathbf{p}_{\text{ff}}(x, y, t)|^2}$$

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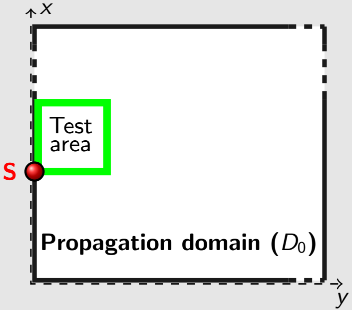

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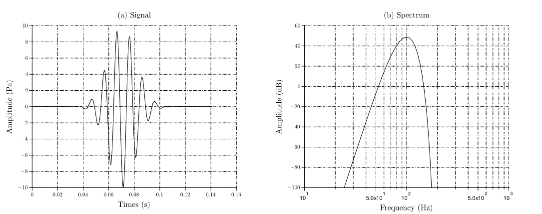

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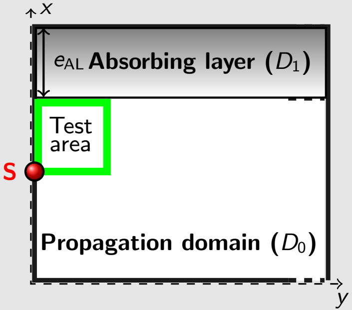

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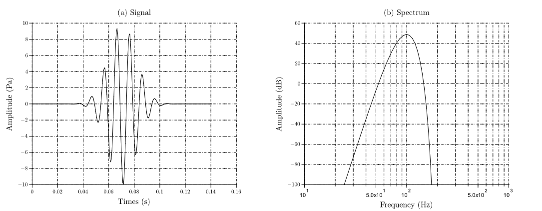
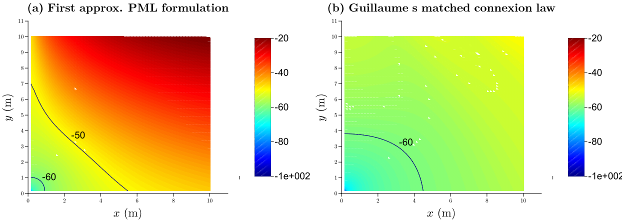


Gaussian impulse signal (a) ; signal spectrum (b).

Mean error results for the first and second set of solutions

- **First set of solutions:**



Mean error function level (dB) inside the test area for an absorbing layer thickness $\mathbf{e}_{AL} = \mathbf{40 nodes}$ ($N_\lambda = 2$, $f = 100$ *Hz*)*: (a)* **first** *approximate PML formulation ; (b) Guillaume's matched connexion law.*

Mean error results for the first and second set of solutions

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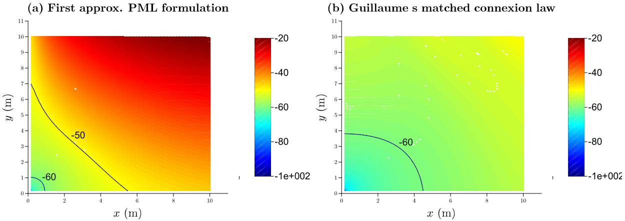
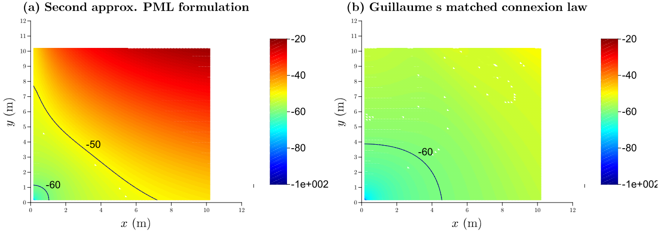


Mean error function level (dB) inside the test area for an absorbing layer thickness $\mathbf{e}_{AL}$ = **40nodes** ($N_\lambda = 2$, $f = 100$ *Hz*): *(a)* **first** *approximate PML formulation ; (b) Guillaume's matched connexion law.*

- **Second set of solutions:**

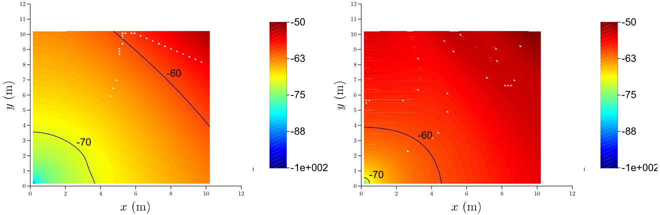


Mean error function level (dB) inside the test area for an absorbing layer thickness $\mathbf{e}_{AL}$ = **40nodes** ($N_\lambda = 2$, $f = 100$ *Hz*): *(a)* **second** *approximate PML formulation ; (b) Guillaume's matched connexion law.*

Mean error results for the third set of solutions

- **Third set of solutions with $e_{AL} = 40$nodes:**

(a) Third approx. PML formulation (b) Guillaume s matched connexion law



Mean error level (dB) inside the test area for an absorbing layer thickness: $\mathbf{e}_{AL} = \mathbf{40nodes} \Leftrightarrow N_\lambda = 2$, $f = 100$ *Hz: (a) third approximate PML formulation ; (b) Guillaume's matched connexion law.*

Mean error results for the third set of solutions

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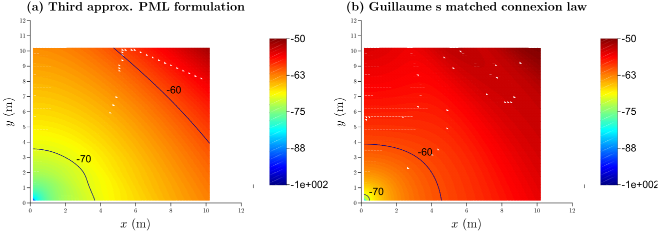
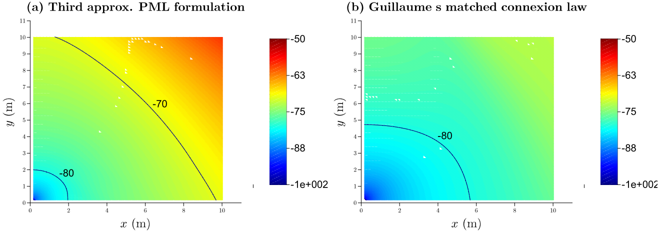


Mean error level (dB) inside the test area for an absorbing layer thickness: $\mathbf{e}_{AL}$ = **40nodes** ⇔ $N_\lambda = 2$, $f = 100$ *Hz: (a) third approximate PML formulation ; (b) Guillaume's matched connexion law.*

- **Third set of solutions with $\mathbf{e_{AL}} = \mathbf{60nodes}$:**



Mean error level (dB) inside the test area for an absorbing layer thickness: $\mathbf{e}_{AL}$ = **60nodes** ⇔ $N_\lambda = 3$, $f = 100$ *Hz: (a) third approximate PML formulation ; (b) Guillaume's matched connexion law.*

Conclusions

From the results:

- **Negative heterogeneity fluctuation** for the first and second approximate PML formulations induces a **mismatching** of the absorbing layer,
- The third formulation is the most efficient for **thin** layers:
  $\Rightarrow$ this approach has been **related to the lossy wave equation** (i.e. simplified PML wave equation).

Outlook:

- Further investigation on the purely dissipative formulation:
  $\Rightarrow$ **split the scattering matrix** to treat only the main propagation axis (Split-Field attenuation),
- **Combination of the purely dissipative network and the empirical matched connexion laws.**

Thanks for your attention

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JTAV 2013 - Blois

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23rd May 2013

J. Hofmann and K. Heutschi, "Simulation of outdoor sound propagation with a transmission line matrix method," *Appl. Acoust.*, vol. 68, pp. 158–172, 2007.

S. El-Masri, X. Pelorson, P. Saguet, and P. Badin, "Vocal tract acoustics using the transmission line matrix (tlm) method," *International conférence on speech and language processing (ICSLP)*, vol. 2, pp. 953–956, 1996.

J. A. Porti and J. A. Morente, "Tlm methods and acoustics," *Int. J. Numer. Model.*, vol. 14, pp. 171–183, 2001.

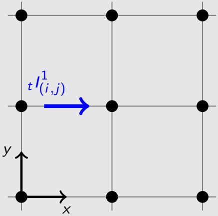
Y. Kagawa, T. Tsuchiya, B. Fuji, and K. Fujioka, "Discrete huygen's model approach to sound wave propagation," *J. Sound Vib.*, vol. 218, pp. 419–444, 1998.

D. de Cogan, W. J. O'Connor, and S. Pulko, *Transmission Line Matrix in Computational Mechanics*. Taylor and Francis, 2006.

G. Guillaume, *Application de la méthode TLM à la modélisation de la propagation acoustique en milieu urbain*. PhD thesis, LCPC, 2009.

Scattering process

Scattering process for 2D transmission line network



$$\mathcal{R} = \frac{Z_T - Z_L}{Z_T + Z_L}$$

$$\mathcal{T} = 1 + \mathcal{R} = \frac{2Z_T}{Z_T + Z_L}$$

- Z_T impedance of the termination,
- Z_L impedance of the line.

- The scattering matrix:

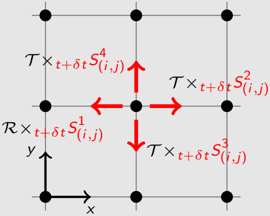
$$ {}_t\mathbf{D}_{(i,j)} = \frac{1}{2}\ {}_t\begin{bmatrix} \mathcal{R} & \mathcal{T} & \mathcal{T} & \mathcal{T} \\ \mathcal{T} & \mathcal{R} & \mathcal{T} & \mathcal{T} \\ \mathcal{T} & \mathcal{T} & \mathcal{R} & \mathcal{T} \\ \mathcal{T} & \mathcal{T} & \mathcal{T} & \mathcal{R} \end{bmatrix}_{(i,j)} . \qquad (6)$$

- Scattering relation:

$$ {}_t\mathbf{S}_{(i,j)} = {}_t\mathbf{D}_{(i,j)}\ {}_t\mathbf{I}_{(i,j)} , \qquad (7)$$

Scattering process

Scattering process for 2D transmission line network



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- Scattering relation:

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Condtion on σ to satisfy the whole system

From equation (5a) the following equality can be written:

$$\eta + \zeta + 4 = 4(1 + \sigma\delta t), \qquad (8)$$

which, combined to equation (5c) is rewriten as:

$$\frac{\eta - \zeta + 4}{\eta + \zeta + 4} = \frac{1 - \sigma\delta t - \sigma^2\delta t^2}{1 + \sigma\delta t}. \qquad (9)$$

If relation (9) is related to equation (5c), then the following condition should be satisfied: $\sigma^2\delta t^2 << 1 - \sigma\delta t$. This induces the following inequality:

$$\frac{\sigma^2\delta t^2}{1 - \sigma\delta t} < \varepsilon, \qquad (10)$$

where ε should be minimized. Thus, the maximum value for the PML attenuation factor σ_{max} is set as a function of the variable ε:

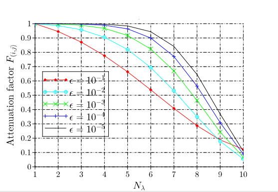
$$\boxed{\sigma < \frac{1}{2\delta t}\left(\sqrt{\varepsilon^2 + 4\varepsilon} - \varepsilon\right)}. \qquad (11)$$

Matched connexion laws

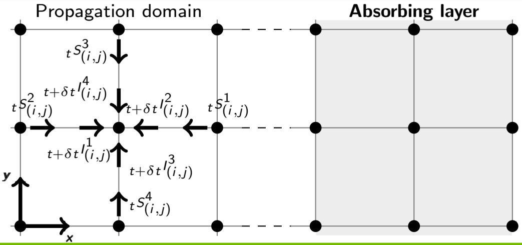
$$\mathbf{F}_{(\mathbf{i},\mathbf{j})} = (1+\epsilon) - \exp\left[-\frac{(x_{(i,j)} - e_{AL})^2}{A}\right]$$

$$A = -\frac{e_{AL}^2}{\ln \epsilon}$$

De Cogan's approach



Guillaume's matched connexion law



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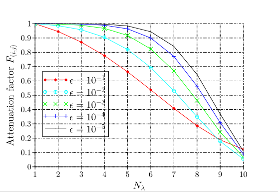


De Cogan's approach

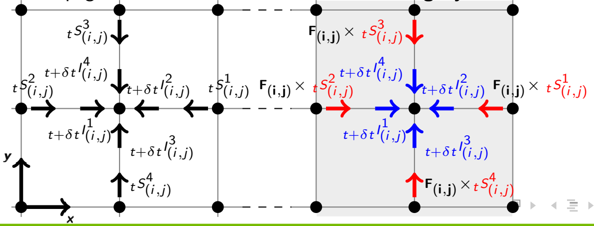
Guillaume's matched connexion law

$$_{t+\delta t}I^1_{(i,j)} = F_{(i,j)} \times {}_tS^2_{(i-1,j)},$$

$$_{t+\delta t}I^2_{(i,j)} = F_{(i,j)} \times {}_tS^1_{(i+1,j)},$$

$$_{t+\delta t}I^3_{(i,j)} = F_{(i,j)} \times {}_tS^4_{(i,j-1)},$$

$$_{t+\delta t}I^4_{(i,j)} = F_{(i,j)} \times {}_tS^3_{(i,j+1)}.$$



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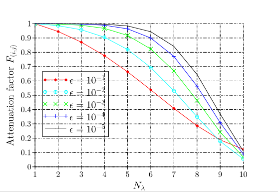

De Cogan's approach

$$_{t+\delta t}I^1_{(i,j)} = \mathbf{F}_{(\mathbf{i},\mathbf{j})} \times {}_tS^2_{(i-1,j)},$$
$$_{t+\delta t}I^2_{(i,j)} = \mathbf{F}_{(\mathbf{i},\mathbf{j})} \times {}_tS^1_{(i+1,j)},$$
$$_{t+\delta t}I^3_{(i,j)} = \mathbf{F}_{(\mathbf{i},\mathbf{j})} \times {}_tS^4_{(i,j-1)},$$
$$_{t+\delta t}I^4_{(i,j)} = \mathbf{F}_{(\mathbf{i},\mathbf{j})} \times {}_tS^3_{(i,j+1)}.$$

Guillaume's matched connexion law

$$_{t+\delta t}I^1_{(i,j)} = \mathbf{F}_{(\mathbf{i},\mathbf{j})} \times {}_tS^2_{(i-1,j)},$$
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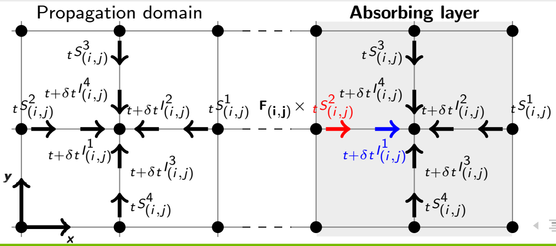


Iterative scheme for the PML wave equation

$$\frac{\partial^2 p(x,y,t)}{\partial t^2} - c_0^2 \left(\frac{\partial^2 p(x,y,t)}{\partial x^2} + \frac{\partial^2 p(x,y,t)}{\partial y^2} \right) + 2\sigma \frac{\partial p(x,y,t)}{\partial t} + \sigma^2 p(x,y,t) = 0$$

Centered finite differences

$$\frac{\partial p(x,y,t)}{\partial t} \sim \frac{p(x,y,t+\delta t) - p(x,y,t-\delta t)}{2\delta t},$$

$$\frac{\partial^2 p(x,y,t)}{\partial t^2} \sim \frac{p(x,y,t+\delta t) - 2p(x,y,t) + p(x,y,t-\delta t)}{\delta t^2},$$

$$\frac{\partial^2 p(x,y,t)}{\partial x^2} \sim \frac{p(x+\delta x,y,t) - 2p(x,y,t) + p(x-\delta x,y,t)}{\delta x^2},$$

$$\frac{\partial^2 p(x,y,t)}{\partial y^2} \sim \frac{p(x,y+\delta y,t) - 2p(x,y,t) + p(x,y-\delta y,t)}{\delta y^2}.$$

TLM notations

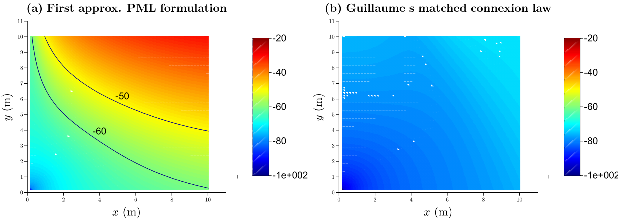
- Node position in a 2D orthogonal mesh: (i,j),
- Time step: δt,
- Spatial step: $\delta x = \delta y = \delta l$,
- Celerity in the TLM network: $c_{\mathrm{TLM}} = c_0 = \sqrt{2}\, c = \sqrt{2}\delta l/\delta t$.

PML propagation scheme

$$\begin{aligned} {}_{t+\delta t}p_{(i,j)} = {} & \frac{1}{2}\,\frac{1}{(1+\sigma\delta t)} \left[{}_t p_{(i+1,j)} + {}_t p_{(i-1,j)} + {}_t p_{(i,j+1)} + {}_t p_{(i,j-1)} - 2\sigma^2 \delta t^2\, {}_t p_{(i,j)} \right] \\ & - \frac{1-\sigma\delta t}{1+\sigma\delta t}\, {}_{t-\delta t}p_{(i,j)} \end{aligned} \qquad (16)$$

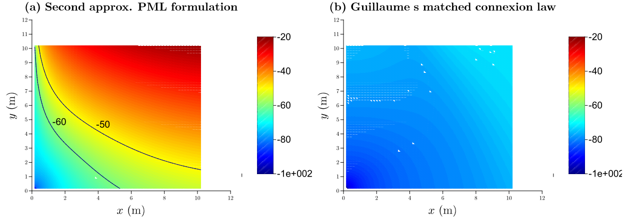
Mean error results for the second set of solutions

- **First set of solutions with $e_{AL} = 60$nodes:**



Mean error function level (dB) inside the test area for an absorbing layer thickness: $e_{AL} = 60nodes \Leftrightarrow N_{\lambda} = 3$: (a) first approximate PML formulation ; (b) Guillaume's matched connexion law.

- **Second set of solutions with $e_{AL} = 60$nodes:**



Mean error function level (dB) inside the test area for an absorbing layer thickness: $e_{AL} = 60nodes \Leftrightarrow N_{\lambda} = 3$: (a) second approximate PML formulation ; (b) Guillaume's matched connexion law.